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Long Cycles in Graphs on a Fixed Surface

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We prove that there exists a function $a: \mathbb{N}_0 \times \mathbb{R}_+ \to \mathbb{N}$ such that

(i) If G is a 4-connected graph of order n embedded on a surface of Euler genus g such that the face-width of G is at least $a(g, \varepsilon)$, then G can be covered by two cycles each of which has length at least $(1-\varepsilon)\, n$.

We apply this to derive lower bounds for the length of a longest cycle in a graph G on any fixed surface. Specifically, there exist functions $b: \mathbb{N}_0 \to \mathbb{N}$ and $c: \mathbb{N}_0 \to \mathbb{R}_+$ such that for every graph G on n vertices that is embedded on a surface of Euler genus g the following statements hold:

(ii) If G is 4-connected, then G contains a collection of at most $b(g)$ paths which cover all vertices of G, and G contains a cycle of length at least $n/b(g)$.

(iii) If G is 3-connected, then G contains a cycle of length at least $c(g)\, n^{\log 2/\log 3}$.

Moreover, for each $\varepsilon > 0$, every 4-connected graph G with sufficiently large face-width contains a spanning tree of maximum degree at most 3 and a 2-connected spanning subgraph of maximum degree at most 4 such that the number of vertices of degree 3 or 4 in either of these subgraphs is at most $\varepsilon\, |V(G)|$.

Key Words: graph theory; surface; face-width; hamilton cycle; circumference.

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1. INTRODUCTION

The notation and terminology in this paper is the same as in [13, 19, 22].

In 1956 Tutte [23] proved that every planar graph G which is not a forest contains a cycle C such that every component of $G-V(C)$ has at most three neighbors on C. We call such a cycle a *Tutte cycle*. Tutte proved that C can be chosen to contain any prescribed edge if G is 2-connected. For a short proof see [18]. Thomas and Yu [17] extended Tutte's theorem to projective planar graphs. It follows that every 4-connected planar or projective planar graph has a Hamiltonian cycle.

This result does not extend to 3-connected planar graphs since there exist planar triangulations on n vertices whose longest cycle is of length $O(n^{\alpha})$, where $\alpha = \log 2/\log 3 \approx 0.63$; cf. [14]. In fact, Grünbaum and Walther [8] conjectured that every 3-connected planar graph of order n contains a cycle of length at least cn^{α} for some positive constant c. Jackson and Wormald [10] proved the existence of a cycle of length at least cn^{β} where c is a positive constant and $\beta \approx 0.2$. Gao and Yu [9] improved their result by showing that every 3-connected planar graph G contains a cycle of length at least $\frac{1}{6}|V(G)|^{0.4}+1$. Recently, Chen and Yu [5] proved the conjecture of Grünbaum and Walther. These results hold also for graphs on the projective plane, the torus, and the Klein bottle [5, 9].

As every graph can be embedded on some surface, these results on Hamiltonian cycles do not generalize to surfaces of higher genera even for 1000-connected graphs. An additional modest condition on the face-width does not help either. Archdeacon *et al.* [1] proved that for each k there exists a k-connected triangulation of an orientable surface having face-width k in which every spanning tree has a vertex of degree at least k. In particular, such graphs are far from being Hamiltonian.

If the surface is fixed and the face-width is large, the situation changes. Thomassen [21] proved that large face-width of a triangulation of a fixed orientable surface implies the existence of a spanning tree of maximum degree at most 4 and that 4 cannot be replaced by 3. It was conjectured in [21] that the additional condition that the triangulation is 5-connected implies that the graph is Hamiltonian, and this was verified by Yu [24]. It was also observed in [21] that "5-connected" cannot be replaced by "4-connected". However, we show in this paper that the cutting technique used in [20, 22] to prove a 5-color theorem for each fixed surface can be used to prove the existence of long cycles in 4-connected or 3-connected graphs on a fixed surface. Specifically, we prove the following theorems.

THEOREM 1.1. *There is a function $a\colon \mathbb{N}_0 \times \mathbb{R}_+ \to \mathbb{N}$ such that for every $\varepsilon > 0$ and every 4-connected graph G that has an embedding of Euler genus g and face-width at least $a(g, \varepsilon)$, there are two cycles C_1, C_2 in G such that*

(1) $V(C_1) \cup V(C_2) = V(G)$, *and*

(2) $|V(C_i)| \geqslant (1-\varepsilon)\,|V(G)|$, *for* $i = 1, 2$.

We apply Theorem 1.1 to prove

THEOREM 1.2. *There exists a function* $b\colon \mathbb{N}_0 \to \mathbb{N}$ *such that, if* G *is a* 4-*connected graph of Euler genus* g, *then* G *contains a collection of paths* $P_1, ..., P_k$, *where* $k \leqslant b(g)$, *which cover all vertices of* G, *and* G *contains a cycle of length at least* $2n/(5b(g))$.

THEOREM 1.3. *There exists a function* $c\colon \mathbb{N}_0 \to \mathbb{R}_+$ *such that, if* G *is a* 3-*connected graph of Euler genus* g, *then* G *has a cycle of length at least* $c(g)\,|V(G)|^{\log 2/\log 3}$.

Barnette [2, 3] proved that every 3-connected planar graph contains a spanning tree of maximum degree at most 3 and a 2-connected spanning subgraph of maximum degree at most 16, and Gao [7] improved the bound 16 to 6 (which is best possible). Sanders and Zhao [16] extended these results to higher surfaces.

Ellingham and Gao [6] modified the method from [21] to prove that large face-width of a 4-connected triangulation on a fixed surface implies the existence of a spanning tree of maximum degree at most 3, and Yu [24] extended this to nontriangulations. The referee informed us that Kawarabayashi, Nakamoto, and Ota [11] showed that such graphs have 2-connected spanning subgraphs of maximum degree 4 and with only few vertices of degree 4.

Theorem 1.1 implies the following extension of Yu's result.

COROLLARY 1.1. *If* $a\colon \mathbb{N}_0 \times \mathbb{R}_+ \to \mathbb{N}$ *is the function defined in Theorem* 1.1 *and* G *is a* 4-*connected graph embedded with face-width at least* $a(g, \varepsilon)$ *on a surface of Euler genus* g, *then* G *contains a spanning tree* T *of maximum degree at most* 3, *a* 2-*connected spanning subgraph* H *of maximum degree at most* 4, *and a path* P *such that*

(a) $P \subseteq T \subseteq H$ *and*

(b) *the number of vertices of degree* 3 *or* 4 *in* T *and in* H *is at most* $2\varepsilon\,|V(G)|$ *and all such vertices are in* $V(P)$.

Proof. Let C_1, C_2 be the cycles in Theorem 1.1. Then we take $H = C_1 \cup C_2$ minus those edges of C_2 which are chords of C_1. Let e be an arbitrary edge of C_1 and let $P = C_1 - e$. Then G has a spanning tree T of maximum degree 3 which is obtained from $H - e$ by deleting only edges in $E(C_2) \setminus E(C_1)$ incident with vertices in C_1. It is obvious that H, T, P have the stated properties. ∎

We shall use the following lemmas.

LEMMA 1.1. *If G is a disconnected graph on a surface S, then $S \backslash G$ contains a simple, closed, twosided curve C which is either noncontractible in S, or contractible in S such that each of* $\mathrm{int}(C)$ *and* $\mathrm{ext}(C)$ *contains a connected component of G.*

Proof. Add on S an edge e between two components G_1, G_2 of G. The facial walk F containing e must contain e twice and in opposite directions because e is a cutedge. Therefore, S has a simple closed twosided curve C (close to $F \cap G_1$) such that $C \cap G = \varnothing$ and C crosses e once. If C is contractible, then $\mathrm{int}(C)$ contains one of G_1, G_2 and $\mathrm{ext}(C)$ contains the other. ∎

LEMMA 1.2. *Let G be a connected graph embedded on a surface S, and let A be a set of vertices such that $G-A$ is disconnected. Then S has a simple closed curve C such that $C \cap G \subseteq A$, and either C is noncontractible or else C is contractible and each of* $\mathrm{int}(C)$ *and* $\mathrm{ext}(C)$ *contains a connected component of $G-A$.*

Proof. Apply the proof of Lemma 1.1 to $G-A$. Let C_0 be the corresponding curve. We may assume that C_0 intersects G only in edges joining a component of $G-A$ and A and that C_0 intersects each such edge at most once and that each such intersection is a crossing. Now we modify C_0 as follows. For each edge $e=uv$ $(v \in A)$ where C_0 intersects G, we replace a short segment of C_0 around that intersection with a simple curve which follows e to v, crosses through v and returns back on the other side of e. The resulting curve C' is homotopic to C_0 and is composed of one or more simple closed curves $C_1, \ldots, C_k$ which intersect G only at A. If all of these curves are contractible, so is C_0. Then each of $\mathrm{int}(C_0)$ and $\mathrm{ext}(C_0)$ contains a connected component of $G-A$ (by the assumption on C_0). It is easy to see (by induction on k) that the same must hold for at least one of the curves $C_1, \ldots, C_k$. ∎

2. PROOF OF THEOREM 1.1

First we introduce some notation.

If G is a plane 2-connected graph with outer cycle C_1 and another facial cycle C_0 disjoint from C_1, then we call G a *cylinder* with *outer cycle* C_1 and *inner cycle* C_0. If H is a graph on a surface of Euler genus g, with disjoint facial cycles C'_0, C'_1 of the same lengths as C_0 and C_1 (respectively), then we can identify C_0 and C'_0 into a cycle C''_0 and identify C_1 and C'_1 into a cycle C''_1. Let M be the graph obtained from the union of G and H after these identifications. The embeddings of G and H determine an embedding of M into a surface of Euler genus $g+2$. We also say that H is obtained from M by *cutting* C_0'' and C_1'' and by *deleting* the cylinder G. The *cylinder-width* of G is the largest integer q such that G has q pairwise disjoint cycles $R_0, ..., R_{q-1}$ such that $C_0 \subseteq \text{int}(R_0) \subseteq \text{int}(R_1) \subseteq \cdots \subseteq \text{int}(R_{q-1})$. The paper [22, Theorem 9.1] has a short proof of the following result: For any natural numbers g and r there exists a natural number $f(g, r)$ such that any 2-connected graph H on $\mathbb{S}_g$ (the orientable surface of Euler genus $2g$) having face-width $\geqslant f(g, r)$, contains g pairwise disjoint cylinders $Q_1, ..., Q_g$ of cylinder-width at least r whose cutting and deletion results in a connected plane graph.

In [22] this was proved for triangulations but the proof extends to all (2-connected) graphs by standard techniques: If H is not a triangulation, we form a triangulation $H_1 \supseteq H$ by adding a new vertex in each face of size at least 4 and joining it to all vertices of H on that face. Then it is easy to see that, if H_1 contains a cylinder of cylinder-width q, then H contains a cylinder of cylinder-width at least $q/2-1$.

Suppose, in addition, that H is 4-connected. Let us focus on one of the g cylinders, say Q_j, and suppose its cylinder-width is $> 10q$. Let $R_0, R_1, ..., R_{10q}$ be the cycles in the definition of the cylinder-width. We select an $i \in \{0, 1, ..., q-1\}$ such that the number of vertices in the sub-cylinder between R_{5i} and R_{5i+5} is smallest possible. Then we cut R_{5i+2} and R_{5i+3} and delete the cylinder between these two cycles. We repeat this procedure for each of the cylinders $Q_1, ..., Q_g$. The resulting graph H' is planar, 2-connected and has therefore a Tutte cycle C containing an edge which is not contained in any of the g cylinders.

We claim that any vertex $v \in V(H) \setminus V(C)$ is in one of the cylinders, say Q_j, and in Q_j, v is between R_{5i} and R_{5i+5}. (In particular, v is on neither R_{5i} nor R_{5i+5}.) To see this, let B be the C-bridge of H' containing v. (That is, B is the component B' of $H'-V(C)$ containing v together with the set A of vertices on C joined to B' and all edges between A and B'.) We apply Lemma 1.2 to the plane graph H' and let Q be the resulting simple closed curve intersecting H' only in A. Now, Q must intersect some face of H' which is not a face of H since otherwise A would separate H (which is impossible since H is 4-connected and $|A| \leqslant 3$) or Q would be noncontractible on $\mathbb{S}_g$ (which is impossible because H has face-width > 3). So we may assume without loss of generality that both C and B intersect R_{5i+2} in some Q_j. Hence C contains at least two vertices of each of R_{5i+1} and R_{5i}. Since B has at most three vertices of attachment, B cannot intersect R_{5i}. So, B is between R_{5i} and R_{5i+2}. Our choice of i (in each of the g cylinders) implies that C misses at most $|V(H)|/q$ vertices of H.

Suppose now that we select the indices i in $\{q, q+1, ..., 2q-1\}$ (one for each of the cylinders). Then we can find another cycle C' in H missing at most $|V(H)|/q$ vertices of H such that $V(C) \cup V(C') = V(H)$. This completes the proof of Theorem 1.1 in the orientable case.

We now turn to the nonorientable case. Let g, q be any natural numbers. Now draw any specific graph H_0 on $\mathbb{N}_g$ (the nonorientable surface of Euler genus g) such that H_0 contains $\lfloor g/2 \rfloor$ pairwise disjoint cylinders of cylinder width $10q+1$ whose removal results in a connected graph in the projective plane (if g is odd) or the sphere (if g is even). Robertson and Seymour [15] proved that, if the face-width of a graph H on $\mathbb{N}_g$ is sufficiently large, then one can delete edges and contract edges of H such that one obtains H_0 on $\mathbb{N}_g$. In particular, H also contains $\lfloor g/2 \rfloor$ pairwise disjoint cylinders of cylinder width $10q+1$ whose removal results in a connected graph in the projective plane or the sphere. If g is even, we repeat the proof in the orientable case. If g is odd, the same proof works, except that we use the extension of Tutte's theorem obtained by Thomas and Yu [17] that every 2-connected graph in the projective plane has a Tutte cycle containing any prescribed edge.

3. PROOF OF THEOREM 1.2

Bondy and Locke [4] proved that if a 3-connected graph has a path of length k, then it has a cycle of length at least $2k/5$. So, it suffices to prove the first statement in Theorem 1.2. We prove this by induction on the Euler genus.

By the theorems of Tutte [23] and Thomas and Yu [17], $b(0) = b(1) = 1$. Suppose that $b(0) \leqslant b(1) \leqslant \cdots \leqslant b(g-1)$ exist. We shall prove that $b(g) \leqslant (4\,a(g, 1/2) + 3g)\, b(g-1) + 100$.

Let G be any 4-connected graph on a surface S of Euler genus $g \geqslant 2$. Let w_0 denote the face-width of G on S. We may assume that $w_0 < a(g, 1/2)$, since otherwise $V(G)$ is covered by two paths by Theorem 1.1.

Consider first the case where $w_0 \geqslant 4$. Let C_0 be a noncontractible simple closed curve intersecting G in w_0 vertices. We think of C_0 as a cycle in the graph obtained from G by adding ($\leqslant$) w_0 edges, and then we cut that graph along C_0. Then C_0 is cut into a cycle C_1, say, and (if C_0 is twosided) a cycle C_2. The resulting graph G_1 is embedded in a surface S' (possibly disconnected) of Euler genus $g-1$ or $g-2$ (if C_0 is onesided or twosided, respectively). We add a new vertex x_1 in the face bounded by C_1 and join it to all vertices of C_1. If C_2 exists, we also add a new vertex x_2 in the face bounded by C_2.

We assume that S' is connected. (The case where S' is disconnected is similar and easier.) We claim that the resulting graph G_1' is 4-connected. Suppose (reductio ad absurdum) that G_1' has a (smallest) vertex set A such that $G_1' - A$ is disconnected and $|A| \leqslant 3$. If A contains x_1, then A also contains two vertices of C_1 by the minimality of A. We now apply Lemma 1.2. It is easy to modify the resulting simple closed curve in S' into a noncontractible curve in S having only a proper subset of $V(C_0)$ in common with G, a contradiction to the definition of the face-width. So, we may assume that A contains neither x_1 nor x_2. Again, we apply Lemma 1.2 and, if necessary, modify the resulting simple closed curve R such that it does not intersect the interior of any of the faces having x_1 or x_2 on the boundary. Then R determines a simple closed curve R' on S such that $R' \cap G \subseteq A$. Since G is 4-connected, R' is noncontractible on S, contradicting the assumption that $w_0 \geqslant 4$. So, G_1' is 4-connected.

By the induction hypothesis, $V(G_1')$ is covered by at most $b(g-1)$ paths in G_1'. After removing x_1, x_2 and some vertices of C_1 (or C_2) from these paths, we obtain at most $2a(g, 1/2)\, b(g-1)$ paths in G which cover $V(G)$.

Consider next the case where $2 \leqslant w_0 \leqslant 3$. We let C_0 be a noncontractible simple closed curve on S intersecting G in at most 3 points. If possible, we choose C_0 such that it is onesided and $|C_0 \cap G|$ is smallest possible subject to that condition. If there are no onesided closed curves intersecting G in $\leqslant 3$ points, then we select a twosided curve C_0 such that $|C_0 \cap G| = w_0$. As in the case $w_0 \geqslant 4$, we think of C_0 as a cycle (of length 2, or 3) and we cut S along C_0 such that C_0 becomes one cycle C_1 of length 4 or 6 (if C_0 is onesided) or two cycles C_1, C_2 of length 2 or 3 if C_0 is twosided. If C_0 is onesided we add a new vertex x_1 and join it to C_1. If C_0 is twosided, we do not add any of x_1, x_2. The resulting graph is called G_1. If G_1 is 4-connected, we apply induction as in the case $w_0 \geqslant 4$. It is easy to see that G_1 is 4-connected if C_0 is onesided. (For, if a separating set A of at most three vertices contains x_1 and two vertices on C_1, then some component of $G_1 - A$ is a path on C_1 and we obtain a contradiction to the minimality of C_1.) Therefore we may assume that C_0 is twosided and that G_1 is not 4-connected. Now we apply Lemma 1.2 where A is a separating vertex set of G_1 with at most three vertices. The resulting simple closed curve C_3 is twosided (otherwise we would have taken that curve as C_0). If necessary, we modify C_3 so that it does not cross C_1 or C_2. (This is possible since $|V(C_0)| \leqslant 3$.) Now we cut C_3 into two cycles C_4 and C_5. If possible, we select a noncontractible curve C_6 in S which does not cross any of C_1, C_2, C_4, C_5 such that C_6 has less than 4 vertices in common with G and we cut C_6 into cycles C_7 and C_8. We continue like this as often as possible. Thus we cut S into surfaces and G into graphs $G_1, ..., G_p$. By Lemma 1.2, each of $G_1, ..., G_p$ is 4-connected or complete. We define an auxiliary multigraph J_1 whose vertices are the graphs $G_1, ..., G_p$. Each of the curves C_{3i} $(i = 0, 1, 2, ...)$ that we have cut along belongs to two (or one) of the graphs $G_1, ..., G_p$, and J_1 will have an edge (or a loop) between these graphs. We say that the curve C_{3i} *corresponds to* that edge of J_1. As G is 4-connected, J_1 has no cutedge.

Next we define a multigraph J_0 with $V(J_0) \subseteq V(J_1)$ as follows. If J_1 is a cycle, we let J_0 consist of a vertex (corresponding to a surface of Euler genus >0 if possible) and a loop. If J_1 is not a cycle, we let J_0 be the unique multigraph without vertices of degree 2 such that J_1 is a subdivision of J_0. Then J_0 has an edge e such that $J_0 - e$ has no cutedge. Let P be the path in J_1 which corresponds to the edge e. If P has length 1, then cutting S and G along the curve corresponding to P results in a 4-connected graph, and we complete the proof by induction. So assume that P has length at least 2. Assume that the notation is such that the first edge of P corresponds to C_0, and the last edge of P corresponds to one of $C_3, C_6, \ldots$, say R.

When we cut C_0 into C_1 and C_2, then S becomes a surface with boundaries C_1 and C_2. If we also cut R into R_1 and R_2, then we disconnect S into surfaces S' and S'' with boundaries C_1, R_1 and C_2, R_2, respectively. We make S', S'' into closed surfaces S'_1 and S''_1 (respectively) by adding a cylinder (handle) with the outer and inner cycle R_1, C_1 and R_2, C_2, respectively. On each of these handles we add edges and possibly one new vertex so that the two graphs on the two handles are either complete graphs with four vertices or 4-connected graphs with 6 vertices (see Fig. 1). Hence the resulting graphs on S'_1 and S''_1 are 4-connected. If these graphs have Euler genus less than g, we complete the proof by induction (similarly as in the case $w_0 \geqslant 4$). So assume that at least one of them has Euler genus g. Hence S' or S'', say S', is a cylinder. By the choice of J_0 and P, S' corresponds to P. To each of S', S'' we add two discs so that C_1, C_2, R_1, R_2 become facial cycles. The graph on S'' is 4-connected or complete and of Euler genus less than g, so we apply induction to that graph. The graph on S' is planar with facial cycles C_1, R_1, each of length 2 or 3. We add a vertex y joined to all vertices of C_1 and a vertex z joined to all vertices of R_1. By [18], the resulting graph M has a path P from y to z such that each P-bridge has at

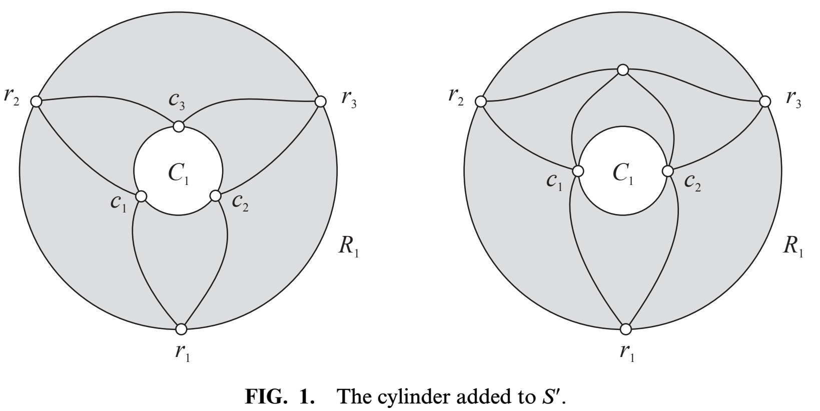

FIG. 1. The cylinder added to S'.

most 3 vertices of attachment. (In [18] it is required that the graph is 2-connected. If M is not 2-connected, we apply [18] to each block of M.) Since G is 4-connected, P contains all vertices of M (except possibly some on C_1 or R_1) and the proof is complete when $2 \leqslant w_0 \leqslant 3$.

Consider finally the case where $w_0 = 1$. In each face which is not bounded by a cycle, we add a vertex joined to all vertices on the boundary. As G is 4-connected, we add at most $3g-3$ new vertices. For, in each augmented face we can draw a simple noncontractible curve having precisely one vertex in common with G, and no two of these curves are homotopic. Hence there are at most $3g-3$ such curves, see, e.g., [12]. Now we repeat the proof of the case when $w_0 \geqslant 2$.

4. PROOF OF THEOREM 1.3

If the Euler genus is at most 2, we apply the result of [5]. For the general case we repeat the inductive proof of Theorem 1.2. The only essential difference is that instead of using [18] at the end of that proof, we apply [5]. Note that we only need to show that G contains a path of length $cn^{\log 2/\log 3}$, by the aforementioned result of Bondy and Locke [4]. We leave the details to the reader.

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